On the numerical evaluation of the anvil force for accurate dynamic stress intensity factor determination

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Abstract

To evaluate precisely the dynamic fracture toughness of a brittle material in the tests with short time-to-fracture, both tup and anvil forces have to be known. Unfortunately, the anvil force is rarely registered by the standard impact testing equipment. The method for numerical evaluation of the support reactions by using registered tup force and the calculated specimen modal parameters is proposed. It assumes that the contact between the specimen and the supports can be described by the quasi-static Hertz’s theory. Both linearized and nonlinear relations for the specimen–support contact compliance are considered. The efficiency of the method has been verified by processing the results of two three-point-bend impact tests reported by Böhm and Kalthoff. The influence of the various calculation parameters (number of eigenmodes taken into account, time step size) and the specimen geometry (length of the specimen overhangs) on the accuracy of determination of the anvil force and dynamic stress intensity factor variation with time is investigated.

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1. Introduction

Instrumented impact fracture testing of precracked beam specimens is widely used for determination of dynamic fracture toughness of brittle materials. During a test, specimen deformation is caused by the tup and anvil loads. However, only the tup force \( F(t) \) is usually registered on standard impact testers. In the absence of the anvil load, to complete the boundary conditions for further theoretical analysis, permanent contact between the specimen and supports is often assumed [1–4]. Unfortunately, this simplistic assumption contradicts experimental observations [5] and leads to noticeable errors in determination of \( K_I(t) \) [6,7].

Unfortunately, simultaneous registration of \( F(t) \) and \( R(t) \) is often not provided by the standard testing equipment. Therefore, an alternative method should be used for \( R(t) \) determination. Such methods are also needed when one wants to reanalyze more precisely previously obtained data in tests in which only \( F(t) \) has been recorded. For these situations \( R(t) \) can be determined numerically. Originally, full-scale 2D finite element (FE) [8] or finite difference [9,10] analyses had been used for accurate \( R(t) \) determination. More simple beam models of the impact specimen utilized for the same purpose usually led to less accurate results [8,10].

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Previously, author of this paper has proposed a modal superposition method (MSM) based technique of \( R(t) \) determination, that combines computational simplicity with the 2D finite element analysis (FEA) level of the accuracy of the results [11]. In this article, an improved version of this method based on more accurate contact law and new algorithms for \( R(t) \) determination will be presented in details together with the analysis of the influence of the various computational parameters on the accuracy of the calculated values of \( R(t) \) and \( K_I(t) \).

2. Theory

The following assumptions have been used to model the specimen interaction with supports:

- specimen material is linearly elastic;
- specimen–supports interaction is modelled by two equal point forces \( R(t) \);
• supports are cylindrically tip-ended with constant radius of curvature \( r \);
• contact between the specimen and supports is described within quasi-static Hertz’s approach.

Let’s consider the 2D model of the specimen (Fig. 1). MSM used in this study predicts ‘global’ (means caused by bending only) displacements of the specimen quite accurately. This method, however, cannot capture local contact displacements effectively. Thus, if the specimen is in contact with the support, MSM predicts undeformed shape of its bottom surface in the contact zone (shown as dashed line in the enlarged part of this zone in Fig. 1). Due to this reason, the depth of the local penetration of the specimen surface by the support \( \delta(R(t)) \) can be defined as

\[
\delta(R(t)) = \begin{cases} 
    u_R(t) & u_R(t) \geq 0 \\
    0 & u_R(t) < 0
\end{cases}
\]

Where positive direction of \( R(t) \) is assumed to be upward, that is, contrary to that for \( F(t) \).

The second part of \( u_R(t) \) is the displacement due to specimen bending \( u_R^{(\text{bend})}(t) \). Using MSM, it can be expressed as

\[
u_R^{(\text{bend})}(t) = \sum_{i=1}^{N_g} \frac{\gamma_i}{\omega_i} \int_0^t F(\tau) \sin(\omega_i(t-\tau))d\tau - 2 \sum_{i=1}^{N_g} \left( \frac{\gamma_i^R}{\omega_i} \right)^2 \int_0^t R(\tau) \sin(\omega_i(t-\tau))d\tau
\]

if \( N_g \) eigenfrequencies and normalized nontrivial symmetrical \(^1\) eigenmodes for the unsupported specimen are taken into account.

Combining the first of Eq. (1) with Eqs. (2) and (3) results in

\[
\delta(R(t)) = \int_0^t F(\tau) \left( \frac{t-\tau}{m} + \sum_{i=1}^{N_g} \frac{\gamma_i^F}{\omega_i} \sin(\omega_i(t-\tau)) \right) d\tau - 2 \int_0^t R(\tau) \left( \frac{t-\tau}{m} + \sum_{i=1}^{N_g} \left( \frac{\gamma_i^R}{\omega_i} \right)^2 \sin(\omega_i(t-\tau)) \right) d\tau
\]

This equation is valid if its right-hand-side is positive, otherwise \( R(t) \equiv 0 \). To solve Eq. (4), a specific form of contact law should be selected. When brittle plastics are tested, both striker and supports can be considered as perfectly stiff. In such a case, the following quasi-static relation between the contact force and the specimen indentation by rigid cylindrical indentor can be used [12]:

\[
\delta(P) = \frac{P}{EB} \left( \varphi(r/W) \left( \ln \frac{\pi EB W^2}{rP} - \frac{v}{1-v} \sqrt{\frac{B}{W}} \right) - 0.47 \right)
\]

Where \( \varphi(r/W) = 0.324 + 0.0179 \sqrt{r/W} \). This equation fits the numerical data obtained by 3D FEA for \( B/W = 0.1, \ldots, 1, r/W = 0.1, \ldots, 0.5, \nu = 0.2, \ldots, 0.4 \) with the accuracy better than 2%. The nonlinearity level of Eq. (5) is weak, thus its linearized form

\[
\delta(P) = \frac{CP}{EB}
\]

\(^1\) Nonsymmetrical eigenmodes are ignored because they do not cause crack opening and nonzero SIF arising.
where
\[
C = \varphi\left(\frac{r}{W}\right) \left( \ln \frac{\pi EBW^2}{rP_{\text{max}}} - \frac{v}{1 - v} \sqrt{\frac{B}{W} + \frac{1}{2}} \right) - 0.47
\]

(7)
can be also used without the essential loss of accuracy [12].

Eq. (4) can be easily solved numerically (see Appendix A for details) both for nonlinear (Eq. (5)) and linearized (Eq. (6)) contact force-indentation relations. When \(F(t)\) and \(R(t)\) are known, \(K_1(t)\) can be calculated using the formula
\[
K_1(t) = k_s^{(1)} \int_0^t F(\tau) \left( \sum_{j=1}^{N_s} \eta_j^{(1)} \cos(\omega_j(t - \tau)) \right) d\tau + k_s^{(2)} \int_0^t R(\tau) \left( \sum_{j=1}^{N_s} \eta_j^{(2)} \cos(\omega_j(t - \tau)) \right) d\tau
\]

(8)
where \(k_s^{(i)}\) are the multipliers with dimensions of SIF per unit force, which are numerically equal to the SIF values for the static one-point bending (1PB) for \(i = 1\) and static two-point bending for \(i = 2\) (see details in [13–15]). Weight coefficients \(\eta_j^{(i)}\), \(i = 1, 2\) are proportional to the contribution of the \(j\)th normalized eigenmode of unsupported specimen into \(k_s^{(i)}\).

Eq. (8) can be considered as a generalization of the Kishimoto et al. formula [16]. When \(F(t)\) and \(R(t)\) are approximated peace-wise linearly (PWL) or by Fourier series, integration in Eq. (8) can be performed analytically and simple formulas in a closed form can be obtained (see Appendix A for details for the PWL case) [13–15]. Numerical values of \(\omega_j\), \(k_s^{(i)}\), \(\eta_j^{(i)}\), \(i = 1, 2\), \(j = 1, 2, 3\) have been determined previously for 2D (plane stress) [13–15] and 3D [17] models of the specimen and fitted polynomially.

3. Calculations

To check the accuracy of the method proposed, results of two impact tests reported by Böhme and Kalthoff [5] have been processed. In these tests large scale Araldite B (\(E = 3.38\) GPa, \(\rho = 1260\) kg/m\(^3\), \(v = 0.33\)) with the same width \(W = 0.1\) m, thickness \(B = 0.01\) m and crack length \(a = 0.03\) m were used. The only difference between these specimens was their length. For one specimen, which will be called ‘long specimen’ or shortly LS, \(L = 0.55\) m, for another ‘short specimen’ or SS) \(L = 0.412\) m that is only a bit longer than the specimen span (\(4W = 0.4\) m). Due to this difference, interaction between the specimen and a support for LS and SS was quite different. SS has relatively long 1PB period of the specimen deformation, which was finished by high velocity impact of the specimen ends against the anvil. This impact caused high amplitude oscillations of \(R(t)\) with the second loss of the contact between the specimen and support (Fig. 2a). In consequence, experimental \(K_1(t)\) determined using caustics method presents noticeable oscillations too (Fig. 2b).
On the other hand, large overhangs in the LS case caused reduction of 1PB period, decreasing of the initial velocity of the specimen/anvil contact and, finally, more smooth $R(t)$ and $K_I(t)$ curves (Fig. 3).

For both tests, DSIF have been determined using plane stress FEA (standard Newmark method has been used for time-stepping procedure) and by numerical solution of Eqs. (4) and (8). FE calculations have been performed for three types of boundary conditions. In all cases experimentally registered $F(t)$ has been used to model specimen/striker interaction. To model specimen/support interaction, however, three different situations have been examined:

1. $R(t)$ is registered experimentally;
2. There are one-side frictionless contact conditions between the specimen and the perfectly stiff cylindrical supports with $r = 0.01$ m. Here numerical $R(t)$ has been determined from the solution of this dynamic contact problem as the total contact force between the specimen and a support;
3. The specimen is in the permanent one-point contact with the supports. Numerical $R(t)$ in this case was equal to the specimen reactions in the contact point.

All FE calculations presented in this article have been performed by commercial code ADINA 7.4. Modal superposition analysis proposed in the previous section, have been carried out using:
(1) Experimentally registered $F(t)$ and $R(t)$ (that means only Eq. (8) has been used to calculate $K_1(t)$);
(2) Experimental $F(t)$ and numerical $R(t)$ obtained from the solution of Eq. (4) for both linearized and nonlinear contact force/contact displacement relations.

Modal parameters of the specimens (namely, $\omega_i$, $\psi_i^F$, $\psi_i^R$, $\eta_i^{(1)}$, $\eta_i^{(2)}$) have been determined by ADINA for the first 19 symmetrical eigenmodes using the same FE meshes as in FEA mentioned above. All MSM-related calculations have been carried out by freeware $DSIFcalc$ [18] program.

4. Results and discussion

4.1. Introductionary remarks

Concept of 1PB is of vital importance for proper interpretation of the results presented in this section. Unfortunately, commonly used definition of 1PB phase as a time when the specimen deformation is caused by the tup force and specimen inertia only is not strict. According to this definition, for the particular point inside the specimen, 1PB stage is finished when the elastic waves generated by the anvil forces arrive into this point. Due to finiteness of the elastic waves velocity, duration of the 1PB stage is different for different
parts of the specimen. To interpret the results of an impact test, two parameters, namely \( t^R_{1PB} \) (it is a 1PB phase duration for the points in the specimen/support contact zone) and \( t^K_{1PB} \approx t^R_{1PB} + t^R_{tr} \) (the same for DSIF), should be considered. If the specimen is in permanent contact with supports \( t_{1PB}^R \equiv t_{tr}^R \) and \( t_{1PB}^K \approx 2t_{tr}^R \), however, for most real-life specimen configurations \( t_{1PB}^R > t_{tr}^R \).

4.2. Finite element analysis results

Results of FEA are compared with the experimental data and the results of MSM computations in Figs. 2 and 3. For both tests, the best agreement between numerical and experimental DSIF has been obtained when experimental \( F(t) \) and \( R(t) \) have been used in the calculation. In the case of dynamic contact problem, computed \( R(t) \) are noticeably different from the experimental anvil forces. In LS case, numerical \( R(t) \) at first is higher that the experimental one. After that, for time interval 0.75–1.35 ms we have the opposite situation. In SS case, disagreement between the computed and experimental anvil forces is observed after the specimen bouncing. Perhaps, the main reason of this disagreement in both cases is the assumption about the perfect stiffness of the supports.

Assumption that the specimen is in permanent contact with the supports can lead to essential differences between the true and computed values of \( R(t) \). The level of such a difference depends on the difference between \( t_{tr}^R \) (about 195 \( \mu s \) for the specimens considered) and \( t_{1PB}^R \). This difference is large for SS case, therefore, for the simply supported SS large negative \( R(t) \) arises (Fig. 2a) and causes artificial drop of \( K_1(t) \) together with its subsequent oscillations out-of-phase with respect to the experimental DSIF (Fig. 2b).

In LS case, \( t_{1PB}^R \) is much smaller, so the initial negative \( R(t) \) for the simply supported specimen is small too (Fig. 3a). For \( t > t_{1PB}^R \), \( R(t) \) determined for simplified specimen/support boundary conditions is very close to the same quantity determined for the exact one-side contact condition. Due to this reason corresponding DSIFs are nearly the same (Fig. 3b). It means that LS can be considered as simply supported one without noticeable drop in accuracy of DSIF determination. In this case Eq. (8) can be substituted by the simpler one (which is, in fact, a simple multimode extension of the original formula of Kishimoto et al. [16])

\[
K_1(t) = k_s^{(3)} \int_0^t F(\tau) \left( \sum_{j=1}^{N_k} \eta_j^{(3)} \cos(\omega_j(t - \tau)) \right) d\tau
\]

(9)

where \( k_s^{(3)} \) is the SIF per unit force for the conventional static three-point bending, \( \omega_j \) is the eigenfrequency of the \( j \)th symmetrical eigenmode of the simply supported specimen, \( \eta_j^{(3)} \) is the weight coefficient proportional to the contribution of the \( j \)th normalized eigenmode into \( k_s^{(3)} \). However, Eq. (9) has two main drawbacks when compared with Eq. (8). First, Eq. (8) is more general and can be used for any geometry configuration of the specimen.

Second, the modal parameters used in Eq. (8) are determined for the unsupported specimen. It means that they are independent of the mechanical parameters of the testing machine (compliances of the striker and the supports, specimen/support and specimen/striker contact compliances, etc.). Thus, modal data for such a specimen should be determined by FEA only once for various combinations of the specimen geometry parameters (like it has been done for DSIFcalc program) and then they can be used for processing of the results of various impact tests.

Considering specimen as a simply supported one implies neglecting the anvil compliance. It can be taken into account by substitution perfectly stiff point supports by elastic springs permanently connected to the specimen. In such a case, however, compliance of these springs will affect modal parameters of the specimen (more precisely, of the system specimen + supports). Thus, these parameters should be determined numerically for each particular pair specimen + supports what is time consuming.

Due to above discussed reasons, utilizing of any form of assumption about the permanent contact between the specimen and supports should be avoided.
4.3. Modal superposition method results

To obtain the highest accuracy, all calculations using MSM that are discussed in this subsection have been performed for \( N_R = N_K = 19, \Delta t_R = \Delta t_K = 0.01T_1 \).

As well as FEA, MSM gives the best agreement between numerical and experimental DSIF when experimental \( F(t) \) and \( R(t) \) have been used in computations. In this case, for both tests considered, FEA and MSM results are practically identical. Due to this reason the latter ones are not shown in Figs. 2b and 3b.

Numerical \( R(t) \) values obtained using nonlinear and linearized force/indentation relations are very close to each other and agree well with the experimental anvil forces for both tests considered (Figs. 2a and 3a). Strictly speaking, these numerical anvil forces should be also very close to \( R(t) \) determined from the FE solution of the dynamic contact problem. Such a closeness is present in the LS results, however, for SS, the anvil forces computed using MSM are much closer to the experimental data than to the FEA result. In consequence, \( K_1(t) \) curves obtained by dynamic contact FEA and MSM are very close in the LS case, whereas they are noticeably different in the SS case and MSM results are more close to the experimental data. Perhaps this phenomenon is caused by the fact that larger number of eigenmodes should be taken into account to obtain \( R(t) \) and, consequently, \( K_1(t) \) accurately in the SS case than in the LS one (see details in Section 4.4). Detailed analysis has shown that the accuracy of time-stepping FEA in both cases corresponds to the accuracy of MSM for \( N_R = 5–6 \). Such a number of eigenmodes is sufficient to obtain good results for LS and is too small for SS.

4.4. Influence of the some computational parameters on the accuracy of MSM calculations

There are two types of parameters that can affect the accuracy of \( R(t) \) and, in consequence, \( K_1(t) \) determination using MSM: number of eigenmodes taken into account (\( N_R \) and \( N_K \)) and time step (\( \Delta t_R \) and \( \Delta t_K \)). Influence of each of this parameters will be considered below separately. Nonlinear form of the force/indentation law has been used in all calculations (results for the linearized contact law were practically the same).

4.4.1. Influence of the number of eigenmodes considered

From Eqs. (A.4) and (A.6), it is easy to see that the contribution of the \( j \)th eigenmode of the specimen into \( R(t) \) can be measured by the absolute values of weight coefficients \( w_{RF}^j = |\psi_f^j \phi^R_j \omega^2_j| \) and \( w_{RR}^j = |(\psi^R_j \phi^R_j \omega^2_j)| \) for the tup and anvil force related terms, respectively. Thus, relative contribution of each higher eigenmode (with respect to the contribution of the first one) into \( R(t) \) can be estimated by the coefficients \( rw_{RF}^j = w_{RF}^j / w_{RF}^1 \) and \( rw_{RR}^j = w_{RR}^j / w_{RR}^1 \). Dependence of this coefficients on the mode number for two specimens considered is presented in Fig. 4. The following observations can be made from this figure:

- The influence of high eigenmodes is more essential for SS than for LS.
- Although the contribution of the third and fourth modes is the highest, influence of the other modes is noticeable too.

Similarly, from Eq. (A.9) it is easy to see that the contribution of \( j \)th eigenmode into \( K_1(t) \) is proportional to \( w_{KF}^j = |\eta_f^j \omega_j| \) and \( w_{KR}^j = |\eta^R_j \omega_j| \) for the tup and anvil forces related terms, respectively. Again, the relative contribution of higher eigenmodes can be presented by the coefficients \( rw_{KF}^j = w_{KF}^j / w_{KF}^1 \) and \( rw_{KR}^j = w_{KR}^j / w_{KR}^1 \). For the specimens considered, even the maximum values of these weight coefficients are approximately twice smaller than the maximum values of the similar coefficients for \( R(t) \) (see Fig. 5). In contrast to the previous case, contributions of the modes starting from the fifth have negligible influence on \( K_1(t) \).
Influence of \( N_R \) on \( R(t) \) for the SS is presented in Fig. 6a. Although there is some ‘stabilization’ in the accuracy of \( R(t) \) for \( N_R \geq 7 \), the convergence, in general, is slow.

Numerical values of \( R(t) \), obtained in such a way for different \( N_R \), together with the experimental \( F(t) \) were used to calculate \( K_I(t) \) by Eq. (8) with fixed \( N_K = 19 \). It is easy to see that for SS the accuracy of \( R(t) \) determination has limited influence on the accuracy of DSIF (Fig. 6b). Nevertheless, to evaluate \( K_I(t) \) precisely, sufficiently large number of eigenmodes should be still taken into account.

In the LS case, \( N_R \) has moderate influence on the accuracy of \( R(t) \) determination (Fig. 7a). Corresponding \( K_I(t) \) (again obtained for fixed \( N_K = 19 \)) are practically independent of \( N_R \) (Fig. 7b), so even very crude one-mode estimation of the anvil force, that clearly fails to represent the IPB-related delay in \( R(t) \) initiation, can be used to obtain accurate \( K_I(t) \).

It is easy to see from Eq. (8) that two factors affect the accuracy of \( K_I(t) \) calculation: accuracy of loading registration or calculation and the number of eigenmodes \( N_K \) taken into account in this formula. The influence of the first factor with respect to the accuracy of \( R(t) \) has been considered above. To check the
influence of the second one, for both tests, $K_I(t)$ has been calculated for different $N_K$ values using experimental $F(t)$ and the most accurate computed $R(t)$ (namely, for $N_K = 19$, $\Delta t_k = 0.01T_1$). The results for SS (Fig. 8a) show that even one-mode estimation of $K_I(t)$ gives sufficiently good results. Results for $N_K = 2, \ldots, 19$ are practically identical. The convergence rate for LS is a bit lower and $K_I(t)$ become nearly the same for $N_K \geq 3$ (Fig. 8b).

To summarize the above discussion, the high influence of the relative length of the specimen overhangs on the computational aspects of $K_I(t)$ determination should be noted. In general, the accuracy of $K_I(t)$ is much less sensitive than the accuracy of $R(t)$ to the number of eigenmodes considered. However, suggestions that taking into account only two first eigenmodes is sufficient to evaluate DSIF accurately, for any specimen configuration [2,4], seems to be too optimistic. To be on safe side, even larger number of eigenmodes should be used (e.g., $N_K = 6$ is the default value in DSIFcalc program), especially for the short time-to-fracture 1PB tests.

4.5. Influence of the time steps length

If is easy to see from Eq. (8) that if $F(t)$ and $R(t)$ are known, $K_I(t)$ can be calculated directly for any instant of time. There is no need to know DSIF values for the preceding time instants for such calculations.
In practice, it means that accuracy of $K_I(t)$ values is independent of time step $\Delta t_r$ used for this calculations. So the time step used in this case should be only small enough to capture DSIF oscillations sufficiently well. To obtain the results presented in this subsection, $\Delta t_K = 0.01T_1$ has been used.

In contrast, $R(t)$ is calculated using the recurrence formula and can depend on $\Delta t_r$. Moreover, even if both $F(t)$ and $R(t)$ are accurate, large $\Delta t_r$ means their PWL approximation used in Eq. (A.9) will be crude and the final accuracy of $K_I(t)$ can be lower.

The above remarks are supported by the results presented in Figs. 9 and 10. For SS the accuracy of $K_I(t)$ is acceptable for $\Delta t_r$ values up to 10%$T_1$. For LS case, due to more smooth $R(t)$ curve, accuracy of $K_I(t)$ remain high even for very large $\Delta t_r = 0.2T_1$.

From the practical point of view, the most important conclusion from these results is that the accuracy of $K_I(t)$ seams to be quite insensitive for $\Delta t_r$ values that are lower than some threshold.

5. Conclusions

(1) A new method for anvil force calculation, using the tup force registered during a three-point-bend impact test and numerically obtained modal parameters of the specimen, has been proposed. The accuracy of the method has been checked by processing the results of two tests reported in literature.
(2) The method can be used for nonlinear and linearized quasi-static contact force/contact displacement relations. The results obtained using both relations are practically the same.

(3) The accuracy of the results obtained using the method proposed is the same as for full-scale 2D dynamic contact FEA.

(4) To obtain anvil force accurately, a large number of eigenmodes should be taken into account. Fortunately, for the specimens with relatively long overhangs, the accuracy of the $R(t)$ does not affect the accuracy of $K_I(t)$. For such specimens even rather rough one-mode estimation of $R(t)$ allows to calculate $K_I(t)$ accurately.

(5) For known $F(t)$ and $R(t)$, at least three first eigenmodes should be taken into account to capture $K_I(t)$ oscillations accurately. Perhaps, this number of eigenmodes could be even higher for the tests with short time-to-fracture.

(6) Sensitivity of calculated $K_I(t)$ to the time step used for anvil force calculation and piece-wise-linear approximation of the registered tup force depends on the type of the specimen considered. Specimens with relatively long overhangs are less sensitive to the time step value than the specimens with short overhangs.

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A.1. Calculation of the anvil force for the linearized force/indentation equation

After substitution of Eq. (6) into Eq. (4) one can obtain the following Volterra integral equation of the second kind:

\[
CR(t) = u_F(t) - 2 \int_0^t R(\tau) \left( \frac{t - \tau}{m} + \sum_{i=1}^{N_p} \frac{\psi_i^R}{\omega_i} \sin(\omega_i(t - \tau)) \right) d\tau
\]

where

\[
u_F(t) = \int_0^t F(\tau) \left( \frac{t - \tau}{m} + \sum_{i=1}^{N_p} \frac{\psi_i^F \psi_i^R}{\omega_i} \sin(\omega_i(t - \tau)) \right) d\tau
\]

is the tup force caused displacements in the specimen/support contact point that can be calculated for any instant of time. Let us approximate \( R(t) \) PWL as follows:

Fig. 9. Influence of the time step \( \Delta t_R \) on the accuracy of \( R(t) \) (a) and \( K_I(t) \) (b) determination for SS.

Appendix A

A.1. Calculation of the anvil force for the linearized force/indentation equation
where $R_i$ are the anvil force values for the $(n_R + 1)$ evenly spaced time points $t_i = i\Delta t_R$, $i = 0, \ldots, n_R$; $t_0$ corresponds to the beginning of the contact between the specimen and the anvil, therefore $R_0 = R(t_0) = 0$. After substitution of Eq. (A.3) for any $t = t_k$, $1 \leq k \leq n_R$ into Eq. (A.1) and some algebra one can obtain

$$R_k = \frac{u_F(t_k) - \frac{2\Delta t_R^2}{m}\sum_{i=1}^{k-1} R_i(k-i) - 4\sum_{i=1}^{k-1} R_i \sum_{j=1}^{N_k} (\psi_j^R)^2 \sin(\omega_j(k-i)\Delta t_R)(1 - \cos(\omega_j\Delta t_R))}{\omega_j^3 \Delta t_R}$$

(A.4)

Of course, $R_k \equiv 0$, if the right-hand-side of Eq. (A.4) is negative.

Linearized contact compliance $C$ in Eq. (A.4) depends on the maximum value of unknown anvil force. As the first estimation of the maximum anvil force, a half of the maximum value of the registered $F(t)$ can be used. After that the anvil force is calculated and its refined maximum value is used to calculate contact compliance more accurately. The convergence of this process is very fast and usually only two to three iterations are needed to calculate the anvil force with the accuracy better than 1%.
A.2. Calculation of the anvil force for the nonlinear force/indentation equation

The same technique can be used for \( R(t) \) calculation in the nonlinear case. Substitution of Eqs. (5) and (A.3) into Eq. (4) gives

\[
R_k \left( A_1 (A_2 - \ln(R_k)) + \frac{\Delta t_R^2}{3m} + 2 \sum_{j=1}^{N_t} \frac{\left( \psi_j^R \right)^2 \sin(\omega_j \Delta t_R)}{\omega_j^2 \Delta t_R} \right) = uF_k - \frac{2 \Delta t_R^2}{m} \sum_{i=1}^{k-1} R_i (k - i) - 4 \sum_{i=1}^{k-1} R_i \sum_{j=1}^{N_t} \frac{\left( \psi_j^R \right)^2 \sin(\omega_j (k - i) \Delta t_R)}{\omega_j^2 \Delta t_R}
\]

where \( A_1 = \phi(r/W)/(EB) \), \( A_2 = \ln(\pi EB W^2/r) - \sqrt{W/B} - (1 - v) - 0.47/\phi(r/W) \). Eq. (A.4) can be obtained from Eq. (A.5) by substitution the linearized contact compliance \( C \) by ‘nonlinear compliance’ \( \overline{C}(R_k) = A_1(A_2 - \ln(R_k)) \). To determine \( R_k \) from Eq. (A.5) the following algorithm was used:

- For each time step the initial guess value \( R_k^{(0)} \) was calculated by linear extrapolation using \( R_{k-1}, R_{k-2} \).
- If the right-hand-side of Eq. (A.5) is positive (otherwise \( R_k = 0 \)), successive approximations of \( R_k \) were calculated for \( l = 1, 2, \ldots \), by the following formula:

\[
R_k^{(l)} = \frac{uF_k - \frac{2 \Delta t_R^2}{m} \sum_{i=1}^{k-1} R_i (k - i) - 4 \sum_{i=1}^{k-1} R_i \sum_{j=1}^{N_t} \frac{\left( \psi_j^R \right)^2 \sin(\omega_j (k - i) \Delta t_R)}{\omega_j^2 \Delta t_R}}{\overline{C}(R_k^{(l-1)}) + \frac{\Delta t_R^2}{3m} + 2 \sum_{j=1}^{N_t} \frac{\left( \psi_j^R \right)^2 \sin(\omega_j \Delta t_R)}{\omega_j^2 \Delta t_R}}
\]

- Iterations were finished when \( |R_k^{(l)}/R_k^{(l-1)} - 1| < \varepsilon \).

For the specimens considered in this article and \( \varepsilon = 0.001 \), only two to four iterations were needed for convergence. Since the only one term in Eq. (A.6) changes during each iteration, calculations are very fast. In practice, there was no difference in execution time for calculations that used linearized or nonlinear contact force–deflection relations.

A.3. DSIF calculation

Although PWL approximation of loading in the form presented by Eq. (A.3) can be utilized to calculate DSIF by Eq. (8), much simpler formulae can be obtained if the following form of PWL approximation is used:

\[
F(t) = \sum_{i=1}^{n_F} (c_i^F - c_{i-1}^F) (t - t_{i-1}^F) H(t - t_{i-1}^F)
\]

\[
R(t) = \sum_{i=1}^{n_R} (c_i^R - c_{i-1}^R) (t - t_{i-1}^R) H(t - t_{i-1}^R)
\]

where \( c_i^F, i = 0, \ldots, n_F \), and \( c_i^R, i = 0, \ldots, n_R \) are the section slopes for PWL approximations of \( F(t) \) and \( R(t) \), respectively (\( c_0^F = c_0^R = 0 \)), \( t_i^F, t_i^R \) are the abscissae of the corresponding nodal points for these approximations, which, in general, should not be evenly spaced. Substitution of Eqs. (A.7) and (A.8) into Eq. (8) gives the following final formula for DSIF calculation:
\[ K_1(t) = k_1^{(1)} F(t) + k_2^{(2)} R(t) - k_1^{(1)} \sum_{i=1}^{n_F} (c_i^F - c_{i-1}^F) H(t - t_{i-1}^F) \left( \sum_{j=1}^{N_R} \frac{\eta_j^{(1)}}{\omega_j} \sin(\omega_j(t - t_{i-1}^F)) \right) \]

\[ - k_2^{(2)} \sum_{i=1}^{n_R} (c_i^R - c_{i-1}^R) H(t - t_{i-1}^R) \left( \sum_{j=1}^{N_R} \frac{\eta_j^{(2)}}{\omega_j} \sin(\omega_j(t - t_{i-1}^R)) \right) \]  

\[ (A.9) \]

References


