MIXED NUMERICAL-ANALYTICAL APPROACH FOR DYNAMIC ONE-POINT-BEND TEST MODELLING

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Abstract. Specimen deformation during a one-point-bend impact test is modelled within 2D linear elasticity framework. Using modal superposition technique, the closed-form solutions for the striker force and dynamic stress intensity factor (DSIF) variation with time are derived. Parameters used in these solutions have been determined using plane stress finite element model of the specimen and approximated polynomially for a wide range of the specimen geometry parameters. The efficiency of the method proposed has been verified by comparison of the analytical results with the experimental data available in the literature and with the results of finite element analysis.

Keywords: dynamic stress intensity factor, one-point-bend impact test, modal superposition.

1. Introduction. One-point bending (1PB) is an inevitable initial stage of any conventional three-point bend (3PB) impact test when specimen deformation is caused by striker loading and material inertia only. During this stage transient dynamic effects dominate in the specimen response so DSIF $K_t(t)$ is not proportional to the striker or tup force $F(t)$. Thus, to evaluate both $F(t)$ and DSIF precisely, some kind of dynamic analysis of the test should be performed. Although a ‘brute-force’ finite element analysis (FEA) can be used for this purpose, more fast and cheap methods for DSIF calculation have been proposed (Kishimoto et al. (1990), Rokach (1997, 2001), Marur (2000)). The aim of this paper is to derive the closed-form solutions for $F(t)$ and DSIF within the framework of two-dimensional linear elasticity.

2. Problem definition and initial assumptions. Let us consider unsupported SEN specimen impacted by the striker at the midspan (see Fig. 1). The following assumptions have been used to model the specimen interaction with a striker:

1. Specimen material is linearly elastic.

Figure 1. Specimen and the scheme of its loading.
2. Specimen-striker interaction is represented by the point contact force $F(t)$.
3. Striker is modelled by a mass-spring system with spring compliance $C_{\text{str}}$.
4. Specimen-striker contact compliance $C_{\text{cont}}$ can be linearized.

Permissibility of the last assumption has been shown previously for the perfectly stiff striker (Rokach 2003). Altogether these assumptions mean that the total approach $\delta(t)$ for the contact pair specimen+striker is

$$\delta(t) = C_{c} F(t)$$  \hspace{1cm} (1)

where $C_{c}=C_{\text{str}}+C_{\text{cont}}$ can be considered as a generalised contact compliance. On the other hand, $\delta(t)$ is the sum of the total striker deformation and $\delta_{\text{cont}}(t)$ (specimen surface penetration by the striker, see Fig.1) and can be defined as

$$\delta(t) = \begin{cases} u_{\text{str}}(t) - u_{F}(t), & u_{\text{str}}(t) \geq u_{F}(t) \\ 0, & u_{\text{str}}(t) < u_{F}(t) \end{cases}$$  \hspace{1cm} (2)

where $u_{\text{str}}(t)$ is the position of the centre of mass of the striker, $u_{F}(t)$ is the $x$-component of the ‘gross’ (means without local contact displacements) specimen displacement in the contact point. The following equation can be obtained for $u_{\text{str}}(t)$

$$u_{\text{str}}(t) = v_{0} t - \frac{1}{M} \int_{0}^{t} F(\tau)(t - \tau) d\tau$$  \hspace{1cm} (3)

where $M$ is the striker mass, $v_{0}$ is the impact velocity. ‘Gross’ specimen displacement in the striker/specimen contact point consists of two parts. The first is the vertical component of specimen movement as a rigid body $u_{F}^{\text{vb}}(t)$ that can be expressed as

$$u_{F}^{\text{vb}}(t) = \frac{1}{m} \int_{0}^{t} F(\tau)(t - \tau) d\tau$$  \hspace{1cm} (4)

where $m$ is the specimen mass. The second part of $u_{\text{str}}(t)$ is the displacement due to the specimen bending $u_{F}^{\text{bend}}(t)$. Using modal superposition method, it can be expressed as

$$u_{F}^{\text{bend}}(t) = \sum_{i=1}^{m} \left(\frac{\psi_{F}}{\omega_{i}}\right)^{2} \int_{0}^{t} F(\tau) \sin(\omega_{i}(t - \tau)) d\tau$$  \hspace{1cm} (5)
where $\omega_i$ is the $i$th nontrivial symmetrical eigenfrequency of the unsupported specimen, $(\psi_i)$ is the vertical component of the corresponding normalized eigenmode in the contact point. Let us consider $u_b^{\text{bend}}(t)$ for the unit harmonic excitation $F(t) = e^{j\omega t}$, where $j$ is the imaginary unit. Steady-state part of this quantity is 

$$ \sum_{i=1}^{\infty} (\psi_i^2 \omega_i^2 e^{j\omega_i t}) / (\omega_i^2 - \omega^2) $$

Assuming $\omega=0$, the bending displacement of the unsupported specimen caused by unit static loading $C_{1PB}$ can be expressed as

$$ C_{1PB} = \sum_{i=1}^{\infty} (\psi_i^2 \omega_i^2) \quad (6) $$

Due to its units (length per force) $C_{1PB}$ will be called ‘1PB static compliance’ henceforth. To understand the physical sense of this quantity, let us consider static 1PB of a beam by unit force that is balanced by uniformly distributed unit mass proportional loading (see Fig.2). Using the expansion theorem, displacements of the beam $u(y)$, with respect to its undeformed shape, can be expanded into series of the beam eigenmodes $\psi_i(y)$.

Since each of these eigenmodes satisfies condition $\int_{L/2}^{L/2} \psi_i(y) dy = 0$, $u(y)$ must satisfy it too. It means that in Fig. 2 areas above and below $u(y)$, with respect to the $y$ axis, must be the same. For a beam positioned in such a way with respect to coordinate system, the value of its displacement at the midspan $u(0)$ is equal to $C_{1PB}$ and can be obtained either by Eq.(6) or simply by solution of this static problem.

Equation (6) can be formally rewritten as

$$ C_{1PB} = C_{1PB} \sum_{i=1}^{\infty} \eta_i^F \quad (7) $$

where $\eta_i^F = (\psi_i^2 \omega_i^2) / (C_{1PB} \omega_i^2)$ is the dimensionless weight coefficient that is proportional to contribution of the $i$th eigenmode into $C_{1PB}$. It is clear that $\sum_{i=1}^{\infty} \eta_i^F = 1$.

Combining Eqs.(1)-(7) results in

$$ C_i F(t) = v_0 t - \int F(\tau) \left( \frac{(t-\tau)\omega_i^2}{m} + C_{1PB} \sum_{i=1}^{\infty} \eta_i^F \omega_i \sin(\omega_i(t-\tau)) \right) d\tau \quad (8) $$

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1 Non-symmetrical eigenmodes are ignored because they do not cause crack opening and non-zero SIF arising
where \( \zeta^2 = 1 + \frac{m}{M} \). This equation is valid if \( u_{se}(t) \geq u_{fs}(t) \), otherwise \( F(t) = 0 \). When \( F(t) \) is known, DSIF can be calculated using the following formula (Rokach (1998))

\[
K_i(t) = k_s \int_0^\infty F(\tau) \sum_{j=1}^{\infty} \eta_j^{s} \omega_j \sin(\omega_j(t - \tau))d\tau
\]

where \( k_s \) is the SIF-compliance (means SIF per unit force) for static 1PB of the specimen with a crack (see Fig.2), \( \eta_j^{s} \) is the weight coefficient that, similarly as \( \eta_j^{f} \), is proportional to contribution of the \( j \)th eigenmode into \( k_s \).

3. Solution of the problem. Equation (8) is the Volterra integral equation of the second kind with convolution type kernel. Such an equation can be solved using Laplace transform technique. Transform \( L[F(t)] \) of \( F(t) \) can be written as

\[
L[F(t)](s) = \frac{v_0}{C_c \left( \Omega^2 \xi^2 + s^2 \left( 1 + \alpha \sum_{j=1}^{N_f} \frac{\eta_j^f \omega_j^2}{s^2 + \omega_j^2} \right) \right)}
\]

where \( \Omega^2 = (C_c m)^{-1} \), \( \alpha = C_{1PB}/C_c \), \( N_f \) is the number of eigenmodes taken into account. For \( N_f = 1 \) the following inverse transform of the above equation can be obtained analytically

\[
F(t) = \frac{v_0}{4C_c A_1 A_2 \omega_1^2} \left\{ \frac{1-(A_2-A_1)^2}{A_2-A_1} \sin((A_2-A_1)\omega_1 t) + \frac{(A_1+A_2)^2-1}{A_1+A_2} \sin((A_1+A_2)\omega_1 t) \right\}
\]

where \( A_1 = (1/2) \sqrt{\alpha \eta_1^f + (\beta \xi - 1)^2} \), \( A_2 = (1/2) \sqrt{\alpha \eta_1^f + (\beta \xi + 1)^2} \), \( \beta = \Omega/\omega_1 \). For \( N_f > 1 \) Eq.(10) can be rewritten as

\[
L[F(t)](s) = \frac{v_0 \prod_{j=1}^{N_f} (s^2 + \omega_j^2)}{C_c \left( \left( \Omega^2 \xi^2 + s^2 \right) \prod_{j=1}^{N_f} (s^2 + \omega_j^2) + \alpha \sum_{j=1}^{N_f} \eta_j^f \omega_j^2 \prod_{k=1,k \neq j}^{N_f} (s^2 + \omega_k^2) \right)}
\]
The expression in the denominator is the polynomial of the order $N_{{f+1}}$ with respect to $s^2$. If we denote its roots as $-r_i^2$ and assume that there are no repeated roots here (this assumption is valid at least for a few smallest roots we are interested in), then, using partial fraction expansion technique, the inverse transform of Eq.(12) can be written as

$$F(t) = \frac{v_0}{C_e} \sum_{i=1}^{N_{f+1}} B_i \sin(r_i t), \quad \text{where} \quad B_i = \frac{\prod_{k=1}^{N_{f+1}} (\omega_k^2 - r_i^2)}{r_i \prod_{k=i,k \neq i}^{N_{f+1}} (r_k^2 - r_i^2)}$$  \hspace{1cm} (13)

Corresponding equations for DSIF can be obtained by simple substitution of Eqs.(11) or (13) into Eq.(9). Alternatively, combining transformed Eq. (9) and Eq.(10) one can obtain the following equation for the transform $L[K_i(t)]$ of $K_i(t)$

$$L[K_i(t)](s) = \frac{v_0 \eta_i^k}{C_e} \sum_{j=1}^{N_k} \frac{\eta_j^k \omega_j^2}{s^2 + \omega_j^2} \left(1 + \frac{\alpha \sum_{i=1}^{N_k} \eta_i^k \omega_i^2}{s^2 + \omega_i^2} \right)^{-1}$$  \hspace{1cm} (14)

where $N_K$ is the number of eigenmodes taken into account for calculation of DSIF in Eq.(9). Inverse transform of the Eq.(14) can be obtain similarly as for Eq.(12) (by the way, both equations have the same denominator). The final form of the solution for DSIF depends on the relation between $N_f$ and $N_K$. The following equations can be obtained for the simplest cases $N_f = N_K = 1$ and $N_K > N_f = 1$, respectively

$$K_i(t) = \frac{v_0 \eta_i^k}{4C_e A_i A_2 \omega_i} \left( \frac{\sin((A_2 - A_i) \omega_i t)}{A_2 - A_i} - \frac{\sin((A_1 + A_i) \omega_i t)}{A_1 + A_2} \right)$$  \hspace{1cm} (15)

$$K_i(t) = \frac{v_0 \eta_i^k}{4C_e A_i A_2 \omega_i} \left[ \frac{1 - (A_2 - A_i)^2}{A_2 - A_i} \sin((A_2 - A_i) \omega_i t) \sum_{j=1}^{N_k} \frac{\eta_j^k}{1 - (A_2 - A_j)^2} \omega_j^2 \right] + \frac{(A_2 + A_i)^2 - 1}{A_1 + A_2} \sin((A_1 + A_2) \omega_i t) \sum_{j=1}^{N_k} \frac{\eta_j^k}{1 - (A_2 + A_j)^2} \omega_j^2$$  \hspace{1cm} (16)
A bit more cumbersome relations can be obtained for more general cases
$N_F \geq N_K$ and $N_K > N_F > 1$, respectively

$$K_f(t) = \frac{V_0 k_{x}}{C_c} \sum_{i=1}^{N_F+1} B \left( \sum_{j=1}^{N_K} \frac{\eta_j^K}{1-r_i^2/\omega_j^2} \right) \sin(r_i t)$$

(17)

$$K_f(t) = \frac{V_0 k_{x}}{C_c} \left[ \sum_{i=1}^{N_F+1} B_i \left( \sum_{j=1}^{N_K} \frac{\eta_j^K}{1-r_i^2/\omega_j^2} \right) \sin(r_i t) + \sum_{i=N_F+1}^{N_K} \omega_i \eta_i^K D_i \sin(\omega_i t) \right]$$

where $D_i = \frac{\prod_{k=1}^{N_F}(\omega_x^2-\omega_i^2)}{\prod_{k=1}^{N_F}(r_k^2-\omega_i^2)}$  

(18)

Equations (15)-(18) are valid if the specimen has contact with the striker (namely, while $F(t)>0$). For relatively long crack and/or stiff striker, specimen can ‘bounce’ and loss this contact. During the initial part of the separation stage between the specimen and the striker, DSIF continues to grow even when $F(t)=0$. Thus, to evaluate maximum DSIF value properly, we need to know it during the separation stage too. If $t_s$ is the instant of separation, DSIF values for $t>t_s$ can be obtained using Eq.(9) in which $t_s$ as the upper limit of integration, is substituted by $t_k$. Corresponding formulae for DSIF can be easily obtained by substitution of Eqs.(11),(13) into modified Eq.(9). In fact, these equations present free vibration of the specimen starting from the time instant $t_s$. For example, in $N_F=N_K=1$ case, such an equation is as simple as

$$K_f(t) = K_f(t_s) \cos(\omega_l (t-t_s)) + \frac{\dot{K}_f(t_s)}{\omega_l} \sin(\omega_l (t-t_s))$$

(19)

where $K_f(t_s)$ and its time derivative $\dot{K}_f(t_s)$ are calculated using Eq.(15). Of course, such equations are valid only from $t=t_s$ up to the end of the separation phase.

4. Calculations. Equations (11),(13),(15)-(18) contain several parameters that should be determined numerically. Previously (Rokach (1998)), values of $k_{o_i}$, $\eta_i^K$, $\omega_i$, $i=1-3$ were determined by 2D FEA for a wide range of the specimen geometry parameters and the Poisson’s’s ratio values of the specimen material and approximated by polynomials. Similar calculations (details will be published elsewhere) performed in this study for plane stress 2D model of the specimen allow to obtain the following relations
\[ C_{1PB} = \frac{-0.5927 + 2.134\lambda - 2.212\lambda^2 + 0.0889\gamma + 0.1052\gamma^2 + 0.02103\gamma^3}{1 - 1.657\lambda - 0.1177\lambda^2 + 0.8094\lambda^3 + 0.00623\gamma} \]  \hspace{1cm} (20)

\[ \eta_i^F = 0.9674 - 0.1459\lambda + 0.6388\lambda^2 - 0.6031\lambda^3 + \gamma^{-1}(-0.7549 - 0.7247\lambda + 1.987\lambda^2 + \gamma^{-1}(0.802 - 1.396\lambda - 0.005015\gamma^{-1})) \]  \hspace{1cm} (21)

\[ \eta_3^F = \frac{0.03826 - 0.4402\lambda + 0.7754\lambda^2 - 0.3405\lambda^3 - 0.009582\gamma + 0.0009132\gamma^2}{1 - 2.401\lambda - 5.505\lambda^2 - 0.3489\gamma + 0.02653\gamma^2} \]  \hspace{1cm} (22)

where \( \gamma = L/W \), \( \lambda = a/W \). These equations fit numerical data with accuracy about 1% for \( \lambda = 0.2-0.7 \) and the following ranges of the specimen relative length: \( \gamma = 3-6 \) for Eq.\((20)\), \( \gamma = 2-6 \) for Eq.\((21)\) and \( \gamma = 4-6 \) for Eq.\((22)\). Values of \( \eta_i^F \) were not fitted, because for \( \gamma = 2-6 \), \( \lambda = 0.2-0.7 \) they are smaller than 0.007 and can be ignored in the calculations.

5. **Numerical examples.** To check the accuracy of the method proposed, a 1PB impact test reported by Böhme and Kalthoff (1982) has been modelled. In this test...
Araldite B \((E=3.38 \text{ GPa}, \rho=1260 \text{ kg/m}^3, \text{ Poissons’s ratio } \mu=0.33)\) specimen with dimensions \(L=412 \text{ mm}, W=100 \text{ mm}, B=10 \text{ mm} \) and crack length \(a=30 \text{ mm} \) was impacted by the steel striker \((M=4.9 \text{ kg})\) with initial velocity \(v_0=1 \text{ m/s} \). In Fig.3 experimentally recorded impact force and DSIF are compared with the theoretical predictions obtained using Eqs.(11), (15), (19) for \(N_F=N_K=1\) and Eqs.(13), (17) for \(N_F=N_K=3\). In these calculations striker was assumed to be perfectly stiff (means \(C_{st}=0\), \(C_{cost}\) for striker’s radius of curvature \(R=8 \text{ mm} \) was computed using the formula recently proposed by the author (Rokach (2003))). It is easy to see that the results obtained for \(N_F=N_K=3\) are sufficiently close to the experimental data.

To compare the accuracy of the method proposed with FEA, 1PB test for Charpy-like specimen \((L/W=5.5, a/W=0.5)\) has been modelled using commercial program ADINA 8.1. During calculations, the perfectly stiff striker with \(R/W=0.2\) was supposed to move with constant velocity \(v_0/c=0.0002\), where \(c=\sqrt{E/\rho}\) (this velocity corresponds to about 1 m/s for steel). In Fig.4 dimensionless results of FEA and analytical solutions are compared. Contrary to the previous case, here the striker load consists of two separated pikes. Theoretical analysis is valid up to the beginning of the second pike. This time interval is quite short, however, it covers the most important for practice initial part of DSIF growth history up to its maximum value. Although, as previously, both single and multimode approaches allow to predict the maximum DSIF value sufficiently well, only in the \(N_F=N_K=3\) case agreement between numerical and analytical solutions is good for all valid time instants.

References


